J. Fluid Mech. (1964), *vol.* 20, *part* 2, *pp.* 177–181
Printed in Great Britain

Generation of secondary vorticity in a stratified fluid

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(Received 26 March 1964)

The theory is presented for the generation of secondary vorticity in a stratified fluid. The analysis is a generalization of original work of Hawthorne (1951). It is shown that for an incompressible, inviscid and non-diffusive flow, a flow-wise vorticity component will be generated in a curved stream when a density gradient exists in the direction of the bi-normal to the streamline.

1. Introduction

Hawthorne (1951) gave a general theory describing the generation of secondary vorticity in a curved flow of non-uniform velocity distribution. A significant feature of this work is its clear separation of the primary purely kinematical result (applicable to steady flow of any incompressible fluid) from the final specialization to inviscid flows. Recently Yih (1960), in theoretical studies of two-dimensional flow of incompressible non-diffusive stratified fluids, introduced, in two-dimensional form, a transformation which greatly facilitated the solution of the inviscid flow equations of the stratified fluid.

It occurred to the writer that Hawthorne's formulation could be applied in conjunction with Yih's transformation to yield results for secondary flow development in a stratified fluid. Such results would seem to be of interest in relation to thermal boundary layers in which the density varies, and to meteorology. It is the purpose of this work to present this generalization of Hawthorne's formulation on the basis of Yih's method. The analysis is restricted to steady inviscid non-diffusive incompressible flow.

2. Transformation of kinetic equations for steady inviscid stratified flow

The equations of motion and continuity for steady incompressible inviscid flow of a stratified fluid are respectively

$$\rho(\mathbf{V}\,.\,\nabla)\mathbf{V} = -\operatorname{grad}(p+\phi), \tag{1}$$

$$\mathbf{V}\,.\operatorname{grad}\rho = 0, \tag{2^1}$$

$$\operatorname{div}\mathbf{V} = 0. \tag{2^2}$$

In equation (1), ϕ is the potential of the body forces per unit volume which are assumed to be conservative. Equation (2^1) implies that the density of an element of fluid does not vary in the flow direction, the flow being non-diffusive.

We introduce Yih's transformation (1960) for the velocity,

$$\mathbf{V}' = (\rho/\rho_0)^{\frac{1}{2}}\,\mathbf{V}. \tag{3}$$

Equations (1) and (2^2) become respectively

$$\rho_0(\mathbf{V}'.\nabla)\,\mathbf{V}' = -\operatorname{grad}(p+\phi), \tag{4}$$

$$\operatorname{div}\mathbf{V}' = 0, \tag{5}$$

and one can define a modified vorticity

$$\mathbf{\Omega}' = \operatorname{curl}\mathbf{V}', \tag{6^1}$$

where

$$\operatorname{div}\mathbf{\Omega}' = 0. \tag{6^2}$$

From equation (3)

$$(\mathbf{V}'.\nabla)\,\mathbf{V}' = \operatorname{grad}\tfrac{1}{2}\mathbf{V}'.\mathbf{V}' - \mathbf{V}'\times\mathbf{\Omega}' = -\rho_0^{-1}\operatorname{grad}(p+\phi), \tag{7}$$

and the modified Lamb vector is expressed in terms of the modified total head U' as

$$\mathbf{V}'\times\mathbf{\Omega}' = \rho_0^{-1}\operatorname{grad}[p+\rho_0\tfrac{1}{2}\mathbf{V}'.\mathbf{V}'+\phi] = \rho_0^{-1}\operatorname{grad}U'. \tag{8}$$

For the inviscid flow under consideration then

$$\operatorname{curl}(\mathbf{V}'\times\mathbf{\Omega}') = 0. \tag{9}$$

3. General kinematic resolution

Let $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ be unit tangent, principal normal and binormal vectors for the streamline. The actual and modified velocities are given by

$$\mathbf{V} = q\mathbf{t}, \tag{10}$$

$$\mathbf{V}' = (\rho/\rho_0)^{\frac{1}{2}}\,\mathbf{V} = (\rho/\rho_0)^{\frac{1}{2}}\,q\mathbf{t} = q'\mathbf{t}. \tag{11}$$

Following Hawthorne we resolve the modified vorticity into its component

$$\xi'\mathbf{t} = (\mathbf{V}'.\mathbf{\Omega}'/q'^2)\,\mathbf{V}' \tag{12}$$

along the streamline and a component $\mathbf{\Omega}'-\xi'\mathbf{t}$ perpendicular to $\mathbf{V}'$ and in the plane of $\mathbf{V}'$ and $\mathbf{\Omega}'$, according to

$$\mathbf{\Omega}' = \xi'\mathbf{t}+(\mathbf{\Omega}'-\xi'\mathbf{t}) = (\mathbf{V}'.\mathbf{\Omega}'/q'^2)\,\mathbf{V}'+\mathbf{V}'\times(\mathbf{\Omega}'\times\mathbf{V}')/q'^2. \tag{13}$$

Taking the divergence of equation (13) and using $\operatorname{div}\mathbf{V}' = 0$, $\operatorname{div}\mathbf{\Omega}' = 0$ and rearranging, one obtains after a little reduction

$$\mathbf{V}'\operatorname{grad}[\mathbf{V}'.\mathbf{\Omega}'/q'^2] = -2(q')^{-4}\mathbf{V}'\times(\mathbf{V}'\times\mathbf{\Omega}').[(\mathbf{V}'.\nabla)\,\mathbf{V}'] - (q')^{-2}\mathbf{V}'\operatorname{curl}(\mathbf{V}'\times\mathbf{\Omega}'). \tag{14}$$

Again

$$\begin{aligned}\mathbf{V}'.\mathbf{\Omega}'/q' &= \mathbf{\Omega}'.\mathbf{t} = \operatorname{curl}\mathbf{V}'.\mathbf{t},\\ &= [\operatorname{grad}(\rho/\rho_0)^{\frac{1}{2}}]\times\mathbf{V}.\mathbf{t}+(\rho/\rho_0)^{\frac{1}{2}}\mathbf{\Omega}.\mathbf{t},\\ &= (\rho/\rho_0)^{\frac{1}{2}}\mathbf{\Omega}.\mathbf{t},\end{aligned}$$

so that

$$\mathbf{V}'\operatorname{grad}[\mathbf{V}'.\mathbf{\Omega}'/q'^2] = q'\partial[\xi/q]/\partial s, \tag{15}$$

where s is the streamline arc co-ordinate and ξ the component of the vorticity in the flow direction. Also

$$-2\mathbf{V}'\times(\mathbf{V}'\times\mathbf{\Omega}').[(\mathbf{V}'.\nabla)\mathbf{V}'] = -2\mathbf{V}'\times(\mathbf{V}'\times\mathbf{\Omega}').\mathbf{a}',$$

where $\mathbf{a}'$ is the modified steady flow acceleration. Then

$$-2\mathbf{V}'\times(\mathbf{V}'\times\boldsymbol{\Omega}').[(\mathbf{V}'.\nabla)\mathbf{V}'] = -2\mathbf{V}'\times(\mathbf{V}'\times\boldsymbol{\Omega}').q'^2\mathbf{n}/r \tag{16}$$

(r being the streamline radius of curvature) since the component $q'^2\mathbf{n}/r$ is the only contributing component of $\mathbf{a}'$.

Substituting (15) and (16) in (14), one obtains for the generation of secondary vorticity in the stream direction

$$\frac{\partial}{\partial s}\left[\frac{\xi}{q}\right] = -\frac{2\mathbf{t}\times(\mathbf{V}'\times\boldsymbol{\Omega}').\mathbf{n}}{q'^2 r} - \frac{\mathbf{t}.\operatorname{curl}(\mathbf{V}'\times\boldsymbol{\Omega}')}{q'^2}. \tag{17}$$

4. Secondary vorticity generation in inviscid incompressible stratified fluid

Equation (17) is purely kinematical. It applies to an under-determined system which is rendered determinate when one substitutes the appropriate expression for the modified Lamb vector $\mathbf{V}'\times\boldsymbol{\Omega}'$ from the kinetic equations.

For the flow under consideration the modified Lamb vector is lamellar (equation (9)) and

$$\frac{\partial}{\partial s}\left(\frac{\xi}{q}\right) = -\frac{2\mathbf{t}\times(\mathbf{V}'.\boldsymbol{\Omega}')}{q'^2 r}.\mathbf{n} = -\frac{2\mathbf{t}\times(\mathbf{t}\times\boldsymbol{\Omega}')}{q' r}.\mathbf{n}. \tag{18}$$

While $\partial(\xi/q)/\partial s$ may be expressed in terms of the spatial gradient of the transformed total head U' through equation (8), the transformed total head lacks the geometrical significance of the normal total head, and it is more profitable to simplify equation (18) directly by expanding the vector product.

The intrinsic form for $\boldsymbol{\Omega}'$ is

$$\boldsymbol{\Omega}' = \operatorname{curl}\mathbf{V}' = q'(\mathbf{t}.\operatorname{curl}\mathbf{t})\mathbf{t} + (\partial q'/\partial b)\,\mathbf{n} + \{(q/r)' - (\partial q'/\partial n)\}\,\mathbf{b} \tag{19}$$

(see, for example, Truesdell 1954), so that

$$\mathbf{t}\times(\mathbf{t}\times\boldsymbol{\Omega}') = -\frac{\partial q'}{\partial b}\mathbf{n} - \left(\frac{q'}{r} - \frac{\partial q'}{\partial n}\right)\mathbf{b},$$

whence

$$\frac{\partial}{\partial s}\left(\frac{\xi}{q}\right) = \frac{2}{rq'}\frac{\partial q'}{\partial b} = \frac{2}{rq}\frac{\partial q}{\partial b} + \frac{1}{r\rho}\frac{\partial \rho}{\partial b}, \tag{20^1}$$

$$= \frac{2\Omega_n}{rq} + \frac{1}{r\rho}\frac{\partial \rho}{\partial b}, \tag{20^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial b}\log(\rho q^2). \tag{20^3}$$

Equations (20) describe the generation of a vorticity component in the stream direction for a curved streamline due to a velocity gradient in the binormal direction (i.e. due to a vorticity component Ω_n in the direction of the curvature) and due to a density gradient in the binormal direction. The first term of (20^1) or (20^2) represents Hawthorne's result; we see now that a density gradient in the binormal direction has an effect similar to that of a binormal velocity gradient.

Consider a steady plane curved flow in (x, y)-planes parallel to the plane $z = 0$, with a velocity and density gradient in the z-direction. Such might represent a skewed boundary-layer flow along a wall $z = 0$, or an atmospheric flow parallel to the ground $z = 0$. For such a flow $\xi = -q\,\partial\theta/\partial z$ where θ is the angle between the streamline direction and some fixed reference line in the plane $z =$ const., and

$$\frac{\partial}{\partial s}\left(\frac{\xi}{q}\right) = -\frac{\partial}{\partial s}\left(\frac{\partial\theta}{\partial z}\right) = -\frac{\partial}{\partial z}\left(\frac{\partial\theta}{\partial s}\right) = \frac{1}{r^2}\frac{\partial r}{\partial z} = \frac{1}{r}\frac{\partial}{\partial z}\log r. \tag{21}$$

Equation (20^3) reduces to the result that the normal inertia force $\rho q^2/r$ must be independent of z, a conclusion which also follows from the absence of a normal pressure gradient in the planes $z =$ const. The streamline radius of curvature thus increases in the z-direction in proportion to the specific kinetic energy of the flow.

One now considers an example in which the density gradient itself is modified by the secondary flow.

5. Approximate analysis of spiral secondary flow induced by density gradient

Consider the case of a plane curved main flow of constant radius of curvature, for example, the flow in a pipe bent in a circular arc. We assume that the velocity is in essence uniform and consider the specific effect of a density gradient in generating secondary circulation.† It is supposed that initially a density gradient exists perpendicular to the plane of the main flow and its curvature. The density gradient creates a secondary circulation which in turn modifies the density gradient. The centrifugal force tries to sweep the denser fluid to the outside of the curve and the reaction of the re-orientated density gradient results in a 'pendulum-like' secondary circulation about the main streamline.‡

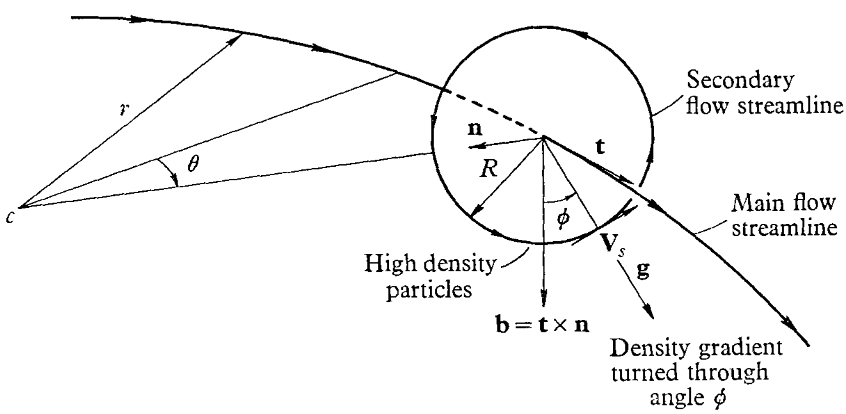

FIGURE 1. Secondary circulation about curved flow induced by density gradient showing rotation of density gradient vector by secondary flow.

† The writer is very grateful to Professor Howard W. Emmons for pointing out to him this particular application of the general formulation.

‡ Due to the close analogy (equations (20)) between secondary vorticity generation from a binormal density gradient and secondaryvorticity generation from a binormal velocity gradient, the analysis and the results obtained in this section are necessarily quite similar to those obtained by Hawthorne (1951).

We assume that the secondary flow circulation $\mathbf{V}_s$ occurs in planes perpendicular to the main streamline as shown in figure 1. Suppose that initially there exists a density gradient

$$\mathbf{g} = \left(\frac{\partial}{\partial b}\log\rho\right)_0 \mathbf{b} = g_0\mathbf{b}. \tag{22}$$

One considers the secondary motion of the high density fluid particles. As a first approximation one may suppose that at time t the secondary flow has rotated the density gradient vector $\mathbf{g}$ through an angle ϕ as shown. Assuming the secondary velocity to be small in comparison to the main flow velocity, the magnitude of the density gradient along the binormal to the main flow is now $g_0 \cos\phi$.

The secondary circulation V_s is related to the secondary vorticity by

$$V_s = \tfrac{1}{2}R\xi, \tag{23}$$

where R is the radius of secondary flow path of the high density particles.

The main and secondary velocities may be written as $q = r\dot{\theta}$, $V_s = R\dot{\phi}$, while the elementary arc of the circular main flow streamline is $rd\theta$.

The general result

$$\frac{\partial}{\partial s}\left(\frac{\xi}{q}\right) = \frac{1}{r}\frac{\partial}{\partial b}\log\rho \tag{20^4}$$

gives

$$\frac{d}{rd\theta}\left(\frac{2}{r}\frac{d\phi}{d\theta}\right) = \frac{g_0}{r}\cos\phi, \tag{24^1}$$

or, approximately,

$$\frac{d^2\phi}{d\theta^2} = m^2\cos\phi, \tag{24^2}$$

where

$$m^2 = \tfrac{1}{2}g_0 r. \tag{24^3}$$

Equation (24^2) is analogous to the equation of motion of a pendulum making an angle ϕ with the horizontal. The secondary circulation will oscillate between $\phi = 0$ and $\phi = \pi$ about the main flow streamline, the main flow deflexion for a complete cycle being given by

$$\Theta = 2\pi/m = 2\pi(2/g_0 r)^{\frac{1}{2}}. \tag{25}$$

The period is thus inversely proportional to the root of the density gradient g_0, and inversely proportional to the root of the main flow radius r. The secondary circulation will pass through zero after each $\pi(2/g_0 r)^{\frac{1}{2}}$ radians of turn.

This work is part of a research programme on secondary flow development in curved flow sponsored by the U.S. National Science Foundation. This work is dedicated to M.J.S.

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